

Various analytic observations on combinations*

Leonhard Euler

1. Let a series, either finite or infinite, of quantities be given to us, as

$$a, b, c, d, e, f, g, h \quad \text{etc.};$$

the letters denote these quantities, which may be either equal or unequal to each other. At the same time, however, I will speak of quantities indicated by different letters as being unequal to each other, even if in examples equal numbers can be substituted in place of them.

2. Now first let new series be formed from these quantities by taking powers, with the sums designated by capital letters A, B, C, D etc. as follows; thus:

$$\begin{aligned} A &= a + b + c + d + e + \text{etc.}, \\ B &= a^2 + b^2 + c^2 + d^2 + e^2 + \text{etc.}, \\ C &= a^3 + b^3 + c^3 + d^3 + e^3 + \text{etc.}, \\ D &= a^4 + b^4 + c^4 + d^4 + e^4 + \text{etc.}, \\ E &= a^5 + b^5 + c^5 + d^5 + e^5 + \text{etc.} \\ &\text{etc.} \end{aligned}$$

These series will be infinite if the number of quantities a, b, c, d etc. has been assumed to be infinite; on the other hand if the number of these quantities is finite and determinate, put $= n$, then all these series will be comprised of that same number of terms.

3. Next let us now form series by taking products of unequal terms from the assumed quantities a, b, c, d etc. in the following way. Namely, first the single quantities are summed, then the products of two unequal terms, from three unequal terms, from four unequal terms, and so on; and let us indicate these

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series with the Greek letters $\alpha, \beta, \gamma, \delta$ etc., so that it follows:

$$\begin{aligned}
\alpha &= a + b + c + d + \text{etc.}, \\
\beta &= ab + ac + ad + ae + bd + \text{etc.}, \\
\gamma &= abc + abd + abe + bcd + \text{etc.}, \\
\delta &= abcd + abce + bcde + \text{etc.}, \\
\epsilon &= abcde + \text{etc.} \\
&\text{etc.}
\end{aligned}$$

If the number of assumed quantities a, b, c, d etc. were infinite, then these series would not only all extend infinitely, but also, the number of forms of these series would be infinite. But if instead the number of the quantities a, b, c, d etc. is finite, put $= n$, then the series α will contain n terms, the second series β will be comprised from $\frac{n(n-1)}{1 \cdot 2}$ terms, the third γ from $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$ terms, the fourth δ from $\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$ terms, and so on, until finally a series is reached that is comprised from a single term, and then all the subsequent series would vanish, as they could not have any terms. It is also clear that the number of series that occur here is $= n$, the last of which consists of a single term, which is the product of all the assumed quantities a, b, c, d, e etc.

4. Since here we only took products from unequal quantities and formed the series treated above from them, by thus repeating the same quantities in products, new series of products from one, two, three, four etc. will be obtained, in which equal factors are not excluded as before; so the series will be obtained thus:

$$\begin{aligned}
\mathfrak{A} &= a + b + c + d + e + \text{etc.}, \\
\mathfrak{B} &= a^2 + ab + b^2 + ac + bc + c^2 + \text{etc.}, \\
\mathfrak{C} &= a^3 + a^2b + ab^2 + b^3 + a^2c + abc + \text{etc.}, \\
\mathfrak{D} &= a^4 + a^3b + a^2b^2 + a^2bc + abcd + \text{etc.}, \\
\mathfrak{E} &= a^5 + a^4b + a^3b^2 + a^3bc + a^2bcd + \text{etc.} \\
&\text{etc.};
\end{aligned}$$

in other words, these series contain all the quantities which can be produced from the multiplication of the given quantities a, b, c, d etc. Again we should note that if the number of quantities a, b, c, d etc. were finite, $= n$, then the first series \mathfrak{A} must have n terms; while the second \mathfrak{B} would have $\frac{n(n+1)}{1 \cdot 2}$ terms, the third \mathfrak{C} would have $\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$ terms, the fourth \mathfrak{D} indeed $\frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}$ terms, and so on.

5. The three orders of series, which we have composed from the given quantities a, b, c, d etc. in three ways, are connected to each other, such that with the order of any one of the series known, then from it the orders of the remaining two can be determined. For the details of how this law works and a method for

investigating this, observation and induction are typically applied for the most part; first noticing of course that $A = \alpha = \mathfrak{A}$, and for the remaining it has been checked that:

$$\begin{aligned}\alpha &= A, \\ \beta &= \frac{\alpha A - B}{2}, \\ \gamma &= \frac{\beta A - \alpha B + C}{3}, \\ \delta &= \frac{\gamma A - \beta B + \alpha C - D}{4}, \\ \epsilon &= \frac{\delta A - \gamma B + \beta C - \alpha D + E}{5} \\ &\text{etc.}\end{aligned}$$

similarly

$$\begin{aligned}\mathfrak{A} &= A, \\ \mathfrak{B} &= \frac{\mathfrak{A}A + B}{2}, \\ \mathfrak{C} &= \frac{\mathfrak{B}A + \mathfrak{A}B + C}{3}, \\ \mathfrak{D} &= \frac{\mathfrak{C}A + \mathfrak{B}B + \mathfrak{A}C + D}{4}, \\ \mathfrak{E} &= \frac{\mathfrak{D}A + \mathfrak{C}B + \mathfrak{B}C + \mathfrak{A}D + E}{5} \\ &\text{etc.}\end{aligned}$$

and also

$$\begin{aligned}\mathfrak{A} &= \alpha, \\ \mathfrak{B} &= \alpha\mathfrak{A} - \beta, \\ \mathfrak{C} &= \alpha\mathfrak{B} - \beta\mathfrak{A} + \gamma, \\ \mathfrak{D} &= \alpha\mathfrak{C} - \beta\mathfrak{B} + \gamma\mathfrak{A} - \delta, \\ \mathfrak{E} &= \alpha\mathfrak{D} - \beta\mathfrak{C} + \gamma\mathfrak{B} - \delta\mathfrak{A} + \epsilon \\ &\text{etc.}\end{aligned}$$

By means of these relations, given the the sums of the series in any one class, the sums of the series contained in the other two classes will be able to be defined.

6. By carefully studying the nature and qualities of these series, the truth of this mutual relation will indeed easily be clear by observation and induction. Truly though, however much we are convinced by the truth of this connection, it will be fruitful to consider the entire problem in the following way; whence at once other properties are offered to us in addition which induction alone does not easily present a path to. Namely, assuming as given the quantities

$$a, b, c, d, e \quad \text{etc.}$$

from which are formed the three classes of series detailed above, let us consider this expression

$$P = \frac{az}{1-az} + \frac{bz}{1-bz} + \frac{cz}{1-cz} + \frac{dz}{1-dz} + \frac{ez}{1-ez} + \text{etc.},$$

and with all the terms resolved into geometric progressions in the usual way this gives

$$\begin{aligned} P = & +z(a+b+c+d+e+\text{etc.}) \\ & +z^2(a^2+b^2+c^2+d^2+e^2+\text{etc.}) \\ & +z^3(a^3+b^3+c^3+d^3+e^3+\text{etc.}) \\ & +z^4(a^4+b^4+c^4+d^4+e^4+\text{etc.}) \\ & \text{etc.;} \end{aligned}$$

all these series are contained in the first class. Then if the sums given above (§2) are written in place of these, it will be

$$P = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.},$$

and thus the sum of this series will be, as we assumed,

$$P = \frac{az}{1-az} + \frac{bz}{1-bz} + \frac{cz}{1-cz} + \frac{dz}{1-dz} + \text{etc.}$$

Also in a similar way, if

$$Q = \frac{az}{1+az} + \frac{bz}{1+bz} + \frac{cz}{1+cz} + \frac{dz}{1+dz} + \text{etc.},$$

by series of the first class it will be

$$Q = Az - Bz^2 + Cz^3 - Dz^4 + Ez^5 - \text{etc.}$$

7. Let us next consider this expression

$$R = (1+az)(1+bz)(1+cz)(1+dz)(1+ez) \text{ etc.};$$

if the factors are actually multiplied into each other and the terms are disposed according to the exponents of z , the coefficient of z will be equal to the sum of the given quantities a, b, c, d etc. The coefficient of z^2 will be the aggregate of all the products of two unequal terms, the coefficient of z^3 will be the aggregate of all products of three unequal terms, and so on; from this it follows that

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5 + \text{etc.}$$

according to the definitions given above (§3).

While if we put

$$S = (1 - az)(1 - bz)(1 - cz)(1 - dz)(1 - ez) \text{ etc.},$$

it will be just by making z negative

$$S = 1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \epsilon z^5 + \text{etc.}$$

8. In order to compare these series R and S with the preceding P and S , it should be noted that

$$lR = l(1 + az) + l(1 + bz) + l(1 + cz) + l(1 + dz) + \text{etc.},$$

which by taking the differentials will be

$$\frac{dR}{Rdz} = \frac{a}{1 + az} + \frac{b}{1 + bz} + \frac{c}{1 + cz} + \frac{d}{1 + dz} + \text{etc.},$$

which multiplied by z gives the same previous expression which above we called Q , so that it will be

$$Q = \frac{z dR}{R dz}.$$

Also in a similar way it will be

$$\frac{dS}{Sdz} = -\frac{a}{1 - az} - \frac{b}{1 - bz} - \frac{c}{1 - cz} - \text{etc.},$$

from which one obtains

$$P = \frac{-z dS}{S dz}.$$

9. Now, since

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \text{etc.},$$

it will be

$$\frac{z dR}{dz} = \alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + 5\epsilon z^5 + \text{etc.}$$

and hence

$$\begin{aligned} Q &= Az - Bz^2 + Cz^3 - Dz^4 + Ez^5 - \text{etc.} \\ &= \frac{\alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + 5\epsilon z^5 + \text{etc.}}{1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5 + \text{etc.}}. \end{aligned}$$

Then, from the equality of these expressions the following relations between the letters A, B, C, D etc. and $\alpha, \beta, \gamma, \delta, \epsilon$ etc. occur:

$$\begin{aligned} A &= \alpha, \\ \alpha A - B &= 2\beta, \\ \beta A - \alpha B + C &= 3\gamma, \\ \gamma A - \beta B - \alpha C - D &= 4\delta, \\ \delta A - \gamma B + \beta C - \alpha D + E &= 5\epsilon \\ &\text{etc.} \end{aligned}$$

Indeed in a similar way, from the other equation $P = \frac{-z dS}{S dz}$ it follows that

$$\begin{aligned} P &= Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.} \\ &= \frac{\alpha z - 2\beta z^2 + 3\gamma z^3 - 4\delta z^4 + 5\epsilon z^5 - \text{etc.}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \epsilon z^5 + \text{etc.}}, \end{aligned}$$

which even yields the same determinations which we gave above (§5).

10. Also, by integrating the equation $Q = \frac{z dR}{R dz}$ it follows that $\int \frac{Q dz}{z} = lR$. Indeed because $Q = Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}$, it will be

$$\int \frac{Q dz}{z} = Az - \frac{Bz^2}{2} + \frac{Cz^3}{3} - \frac{Dz^4}{4} + \text{etc.},$$

and the value of this series thus expresses the logarithm of this series

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

Therefore because

$$l(1 + \alpha z + \beta z^2 + \gamma z^3 + \text{etc.}) = Az - \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 + \text{etc.},$$

then from the equation $\int \frac{P dz}{z} = -lS$ it will be

$$l(1 - \alpha z + \beta z^2 - \gamma z^3 + \text{etc.}) = -Az - \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 - \text{etc.}$$

So if k is written as the number whose hyperbolic logarithm is = 1, we will have

$$1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.} = k^{Az - \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 + \text{etc.}}$$

and

$$1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.} = k^{-Az - \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 - \text{etc.}}$$

11. Also noteworthy are the reciprocals expressions of R and S , of course $\frac{1}{R}$ and $\frac{1}{S}$. Indeed it is

$$\frac{1}{S} = \frac{1}{(1 - az)(1 - bz)(1 - cz)(1 - dz) \text{etc.}};$$

to express the value of this fraction by a series whose terms proceed according to powers of z , it is clear that all these geometric progressions should be multiplied into each other

$$\begin{aligned} \frac{1}{1 - az} &= 1 + az + a^2 z^2 + a^3 z^3 + a^4 z^4 + \text{etc.}, \\ \frac{1}{1 - bz} &= 1 + bz + b^2 z^2 + b^3 z^3 + b^4 z^4 + \text{etc.}, \\ \frac{1}{1 - cz} &= 1 + cz + c^2 z^2 + c^3 z^3 + c^4 z^4 + \text{etc.}, \\ \frac{1}{1 - dz} &= 1 + dz + d^2 z^2 + d^3 z^3 + d^4 z^4 + \text{etc.} \\ &\text{etc.} \end{aligned}$$

In the product, after the first term 1, the coefficient of z will be the sum of the quantities $a + b + c + d + \text{etc.}$, the coefficient of z^2 will be the sum of the factors from two, not excluding equal factors in the same product, the coefficient of z^3 will be the sum of the factors from two, and so on. We designated these sums of products above (§4) with the letters of the Germanic alphabet $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc. With these letters thus introduced, we will have

$$\frac{1}{S} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \mathfrak{E}z^5 + \text{etc.}$$

and by treating the value of R in a similar way it will be

$$\frac{1}{R} = 1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \mathfrak{E}z^5 + \text{etc.}$$

12. So these series are the reciprocals of those which we defined under the letters R and S above (§7). And because of this it will be

$$1 = (1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.})(1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.})$$

and even

$$1 = (1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.})(1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}).$$

From either one of these follows the same relation between the values of the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. and $\alpha, \beta, \gamma, \delta$ etc.; namely it will be

$$\begin{aligned} \mathfrak{A} - \alpha &= 0, \\ \mathfrak{B} - \alpha\mathfrak{A} + \beta &= 0, \\ \mathfrak{C} - \alpha\mathfrak{B} + \beta\mathfrak{A} - \gamma &= 0, \\ \mathfrak{D} - \alpha\mathfrak{C} + \beta\mathfrak{B} - \gamma\mathfrak{A} + \delta &= 0 \\ &\text{etc.,} \end{aligned}$$

which is the same relation we already gave above (§5).

13. But if we put $\frac{1}{R} = T$ and $\frac{1}{S} = V$, so that

$$T = 1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.}$$

and

$$V = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}$$

it will be

$$\frac{dR}{R} = -\frac{dT}{T} \quad \text{and} \quad \frac{dS}{S} = -\frac{dV}{V}$$

and therefore it will become

$$P = \frac{z dV}{V dz} \quad \text{and} \quad Q = -\frac{z dT}{T dz}.$$

Now since

$$\frac{z dV}{dz} = \mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.}$$

and

$$-\frac{z dT}{dz} = \mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.},$$

by writing the appropriate values from §6 in place of P and Q we will have these equations

$$Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.} = \frac{\mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.}}{1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}}$$

and

$$Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.} = \frac{\mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}}{1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.}},$$

from which the same relation between the letters A, B, C, D etc. and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. follows as we gave above (§5). Namely it will be

$$\begin{aligned} \mathfrak{A} &= A, \\ 2\mathfrak{B} &= \mathfrak{A}A + B, \\ 3\mathfrak{C} &= \mathfrak{B}A + \mathfrak{A}B + C, \\ 4\mathfrak{D} &= \mathfrak{C}A + \mathfrak{B}B + \mathfrak{A}C + D, \\ 5\mathfrak{E} &= \mathfrak{D}A + \mathfrak{C}B + \mathfrak{B}C + \mathfrak{A}D + E \\ &\text{etc.} \end{aligned}$$

14. From the equations given in §12 it follows that

$$l(1 + \alpha z + \beta z^2 + \gamma z^3 + \text{etc.}) = -l(1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \text{etc.})$$

and

$$l(1 - \alpha z + \beta z^2 - \gamma z^3 + \text{etc.}) = -l(1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \text{etc.}).$$

Then by applying these to §10 it will be

$$l(1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \text{etc.}) = -Az + \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 + \frac{1}{4}Dz^4 - \text{etc.}$$

and

$$l(1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \text{etc.}) = Az + \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 + \frac{1}{4}Dz^4 + \text{etc.}$$

And hence by taking k as the number whose logarithm is $= 1$, it will be

$$1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \text{etc.} = k^{-Az + \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 + \frac{1}{4}Dz^4 - \text{etc.}}$$

and

$$1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \text{etc.} = k^{Az + \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 + \frac{1}{4}Dz^4 + \text{etc.}}$$

15. Now if the letters R and S retain the values assumed above (§7), it will be

$$\begin{aligned} 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.} &= R, \\ 1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.} &= \frac{1}{R} \end{aligned}$$

and

$$\begin{aligned} 1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.} &= S, \\ 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.} &= \frac{1}{S}. \end{aligned}$$

From these the following conclusions are deduced

$$\begin{aligned} 1 + \beta z^2 + \delta z^4 + \zeta z^6 + \theta z^8 + \text{etc.} &= \frac{R+S}{2}, \\ \alpha z + \gamma z^3 + \epsilon z^5 + \eta z^7 + \iota z^9 + \text{etc.} &= \frac{R-S}{2}, \\ 1 + \mathfrak{B}z^2 + \mathfrak{D}z^4 + \mathfrak{F}z^6 + \mathfrak{H}z^8 + \text{etc.} &= \frac{R+S}{2RS}, \\ \mathfrak{A}z + \mathfrak{C}z^3 + \mathfrak{E}z^5 + \mathfrak{G}z^7 + \mathfrak{I}z^9 + \text{etc.} &= \frac{R-S}{2RS} \end{aligned}$$

and hence this proportion is obtained

$$\begin{aligned} 1 + \beta z^2 + \delta z^4 + \zeta z^6 + \text{etc.} : \alpha z + \gamma z^3 + \epsilon z^5 + \eta z^7 + \text{etc.} \\ = 1 + \mathfrak{B}z^2 + \mathfrak{D}z^4 + \mathfrak{F}z^6 + \text{etc.} : \mathfrak{A}z + \mathfrak{C}z^3 + \mathfrak{E}z^5 + \mathfrak{G}z^7 + \text{etc.} \end{aligned}$$

Since it is also

$$\begin{aligned} R - 1 &= \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}, \\ 1 - \frac{1}{R} &= \mathfrak{A}z - \mathfrak{B}z^2 + \mathfrak{C}z^3 - \mathfrak{D}z^4 + \text{etc.}, \end{aligned}$$

it will be

$$R = \frac{\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}}{\mathfrak{A}z - \mathfrak{B}z^2 + \mathfrak{C}z^3 - \mathfrak{D}z^4 + \text{etc.}},$$

and in a similar way, because

$$\begin{aligned} 1 - S &= \alpha z - \beta z^2 + \gamma z^3 - \delta z^4 + \text{etc.}, \\ \frac{1}{S} - 1 &= \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.} \end{aligned}$$

it will be

$$S = \frac{\alpha z - \beta z^2 + \gamma z^3 - \delta z^4 + \text{etc.}}{\mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}}.$$

16. Next indeed if as above (§6) we put

$$\begin{aligned} P &= Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}, \\ Q &= Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}, \end{aligned}$$

it will be from paragraph 9

$$\begin{aligned} \alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + \text{etc.} &= QR, \\ \alpha z - 2\beta z^2 + 3\gamma z^3 - 4\delta z^4 + \text{etc.} &= PS \end{aligned}$$

and in a similar way from paragraph 13 we will have¹

$$\begin{aligned} \mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.} &= \frac{P}{S}, \\ \mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.} &= \frac{Q}{R}. \end{aligned}$$

From these the following corollaries are easily derived:

$$\begin{aligned} \frac{\alpha z - 2\beta z^2 + 3\gamma z^3 - 4\delta z^4 + \text{etc.}}{Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}} &= S = \frac{Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}}{\mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.}}, \\ \frac{\alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + \text{etc.}}{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}} &= R = \frac{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}}{\mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}}, \end{aligned}$$

For the letters R and S we have these quintuple values

$$\begin{aligned} R &= 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}, \\ R &= \frac{1}{1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.}}, \\ R &= \frac{\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}}{\mathfrak{A}z - \mathfrak{B}z^2 + \mathfrak{C}z^3 - \mathfrak{D}z^4 + \text{etc.}}, \\ R &= \frac{\alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + \text{etc.}}{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}}, \\ R &= \frac{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}}{\mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}}, \end{aligned}$$

in which putting $-z$ in place of z everywhere yields the values for S . And from varied combinations of these five values, a great number of properties can be elicited which the three orders of our letters A, B, C, D etc., $\alpha, \beta, \gamma, \delta$ etc., $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. hold between each other, which however we will refrain from pursuing now.

17. From this it is quite clear that that we can descend from what has been explained already to more particular cases, and first indeed this infinite geometric progression will be taken for the series of letters a, b, c, d etc.

$$n, n^2, n^3, n^4, n^5, n^6 \text{ etc.};$$

¹Translator: The *Opera omnia*, Series I, Volume 2, p. 168 and the original version in the *Commentarii*, Volume 13, p. 74 both have $+4\mathfrak{D}z^4$ on the second line, which is incorrect.

with these successively introduced into the above formulae we will have:

$$\begin{aligned}
A &= n + n^2 + n^3 + n^4 + n^5 + \text{etc.} = \frac{n}{1-n}, \\
B &= n^2 + n^4 + n^6 + n^8 + n^{10} + \text{etc.} = \frac{nn}{1-nn}, \\
C &= n^3 + n^6 + n^9 + n^{12} + n^{15} + \text{etc.} = \frac{n^3}{1-n^3}, \\
D &= n^4 + n^8 + n^{12} + n^{16} + n^{20} + \text{etc.} = \frac{n^4}{1-n^4} \\
&\text{etc.}
\end{aligned}$$

Now from §6, we will get two values for the letters P and Q , which will be

$$\begin{aligned}
P &= \frac{nz}{1-nz} + \frac{n^2z}{1-n^2z} + \frac{n^3z}{1-n^3z} + \frac{n^4z}{1-n^4z} + \text{etc.}, \\
Q &= \frac{nz}{1+nz} + \frac{n^2z}{1+n^2z} + \frac{n^3z}{1+n^3z} + \frac{n^4z}{1+n^4z} + \text{etc.}
\end{aligned}$$

and then from the values found for the letters A, B, C, D etc., these other values will arise

$$\begin{aligned}
P &= \frac{nz}{1-n} + \frac{n^2z^2}{1-nn} + \frac{n^3z^3}{1-n^3} + \frac{n^4z^4}{1-n^4} + \text{etc.}, \\
Q &= \frac{nz}{1-n} - \frac{n^2z^2}{1-nn} + \frac{n^3z^3}{1-n^3} - \frac{n^4z^4}{1-n^4} + \text{etc.}
\end{aligned}$$

18. Next, from paragraph 7 we will have the following expressions for R and S

$$\begin{aligned}
R &= (1+nz)(1+n^2z)(1+n^3z)(1+n^4z) \text{ etc.}, \\
S &= (1-nz)(1-n^2z)(1-n^3z)(1-n^4z) \text{ etc.}
\end{aligned}$$

These factors actually multiplied into each other and arranged according to dimensions of z yield these series for R and S

$$\begin{aligned}
R &= 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}, \\
S &= 1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.},
\end{aligned}$$

where the letters $\alpha, \beta, \gamma, \delta$ etc. are thus determined from the supposed series $n, n^2, n^3, n^4, n^5, n^6, n^7$ etc., so that it will be:

I. α = the sum of all the terms; whence it will be

$$\alpha = n + n^2 + n^3 + n^4 + n^5 + n^6 + n^7 + \text{etc.},$$

which is supposed to be a geometric progression, in which each power of n occurs and has the coefficient +1.

II. β = the sum of the factors from two terms; whence it will be

$$\beta = n^3 + n^4 + 2n^5 + 2n^6 + 3n^7 + 3n^8 + 4n^9 + 4n^{10} + \text{etc.},$$

in which series after the third power all the following powers of n occur; moreover, each power occurs as often as it can be made by multiplying two terms of the series α . Since however the multiplication of powers consists in the addition of exponents, the coefficient of any power of n will appear in the series β in as many ways as the exponent of n can be distributed into two unequal parts, or as many ways as this exponent n can be produced from the addition of two unequal integral numbers. Thus the coefficient of the tenth power n^{10} is 4, because 10 can be distributed into two unequal parts in four ways, namely,

$$\begin{aligned} 10 &= 1 + 9, & 10 &= 3 + 7, \\ 10 &= 2 + 8, & 10 &= 4 + 6. \end{aligned}$$

III. γ = the sum of the factors from three unequal terms of the series α ; whence it will be

$$\gamma = n^6 + n^7 + 2n^8 + 3n^9 + 4n^{10} + 5n^{11} + 7n^{12} + 8n^{13} + \text{etc.},$$

in which after the sixth power, all the following powers of n occur. Moreover the coefficient of each power indicates how many ways the exponent can be distributed into three unequal parts, or as often as the same exponent can be produced from the addition of three mutually unequal integral numbers. Thus the power n^{12} has the coefficient 7, because the exponent 12 can be partitioned into three unequal parts in seven ways, as

$$\begin{aligned} 12 &= 1 + 2 + 9, & 12 &= 1 + 5 + 6, \\ 12 &= 1 + 3 + 8, & 12 &= 2 + 3 + 7, \\ 12 &= 1 + 4 + 7, & 12 &= 2 + 4 + 6, \\ 12 &= 3 + 4 + 5. \end{aligned}$$

IV. δ = the sum of the factors from four mutually unequal terms of the series α ; whence it will be

$$\delta = n^{10} + n^{11} + 2n^{12} + 3n^{13} + 5n^{14} + 6n^{15} + 9n^{16} + \text{etc.},$$

whose first power is n^{10} , whose exponent is clearly $1 + 2 + 3 + 4$, or the fourth trigonal number. Each of the following powers appears as often as its exponent can be made from the addition of four mutually unequal integral numbers. Thus the sixteenth power n^{16} has the coefficient 9, because 16 can be distributed into four mutually unequal parts in nine ways. These nine partitions are

$$\begin{aligned} 16 &= 1 + 2 + 3 + 10, & 16 &= 1 + 3 + 4 + 8, \\ 16 &= 1 + 2 + 4 + 9, & 16 &= 1 + 3 + 5 + 7, \\ 16 &= 1 + 2 + 5 + 8, & 16 &= 1 + 4 + 5 + 6, \\ 16 &= 1 + 2 + 6 + 7, & 16 &= 2 + 3 + 4 + 7, \end{aligned}$$

$$16 = 2 + 3 + 5 + 6.$$

The same kind of thing happens for the values of the following letters ϵ, ζ, η etc., which will be

$$\begin{aligned}\epsilon &= n^{15} + n^{16} + 2n^{17} + 3n^{18} + 5n^{19} + 7n^{20} + 10n^{21} + \text{etc.}, \\ \zeta &= n^{21} + n^{22} + 2n^{23} + 3n^{24} + 5n^{25} + 7n^{26} + 11n^{27} + \text{etc.}, \\ \eta &= n^{28} + n^{29} + 2n^{30} + 3n^{31} + 5n^{32} + 7n^{33} + 11n^{34} + \text{etc.}, \\ &\text{etc.}\end{aligned}$$

In all these series, the coefficient of each power of n indicates how many different ways the exponent of n can be resolved into as many unequal parts as the series is numbered from the first. In other words, the coefficient of any term indicates how many ways the exponent of n can be made from the addition of as many mutually unequal integral numbers as the position of the series from which the term is taken is numbered starting at α . Thus in the seventh series the coefficient of the power n^{34} is 11, because the number 34 can be distributed into seven unequal parts in seven ways; these distributions are

$$\begin{aligned}34 &= 1 + 2 + 3 + 4 + 5 + 6 + 13, \\ 34 &= 1 + 2 + 3 + 4 + 5 + 7 + 12, \\ 34 &= 1 + 2 + 3 + 4 + 5 + 8 + 11, \\ 34 &= 1 + 2 + 3 + 4 + 5 + 9 + 10, \\ 34 &= 1 + 2 + 3 + 4 + 6 + 7 + 11, \\ 34 &= 1 + 2 + 3 + 4 + 6 + 8 + 10, \\ 34 &= 1 + 2 + 3 + 4 + 7 + 8 + 9, \\ 34 &= 1 + 2 + 3 + 5 + 6 + 7 + 10, \\ 34 &= 1 + 2 + 3 + 5 + 6 + 8 + 9, \\ 34 &= 1 + 2 + 4 + 5 + 6 + 7 + 9, \\ 34 &= 1 + 3 + 4 + 5 + 6 + 7 + 8.\end{aligned}$$

And indeed from these, the nature of the series which appear for the letters $\alpha, \beta, \gamma, \delta$ etc. is easily seen.

19. Therefore for investigating how many different ways a number can be distributed into a given number of unequal parts, the series expressing the letters $\alpha, \beta, \gamma, \delta$ etc. can be formed, although this work will be exceedingly tiresome. In turn, however, assuming these series as known and already formed, a not inelegant problem can be solved, which was thus proposed to me by the Insightful Naudé:

To define how many ways a given number can be produced from the addition of several integral numbers, mutually unequal, the number of which is given.

The most Insightful Proposer searched thus for how many different ways the number 50 can arise from the addition of seven unequal integral numbers.

For resolving this question it is clear that the appropriate series to consider is η , in which the coefficient of any term indicates how many different ways the exponent of n can be resolved into 7 unequal parts. Hence the series

$$\eta = n^{28} + n^{29} + 2n^{30} + 3n^{31} + 5n^{32} + 7n^{33} + 11n^{34} + \text{etc.}$$

should be continued to the term at which the fiftieth power of n is contained, whose coefficient, which will be 522, shows that the number 50 can be produced in altogether 522 different ways from the addition of seven mutually unequal integral numbers. Thus it is clear that if a convenient and simple way were obtained of forming these series $\alpha, \beta, \gamma, \delta$ etc., Naudé's problem would be brought to a most perfect solution.

20. Thus since a way was given above (§5 and 9) for finding the values of the letters $\alpha, \beta, \gamma, \delta$ etc. from the known values of the letters A, B, C, D etc., for the present question we can easily obtain a solution, since from §17 we have the known values A, B, C, D etc.; and thus it follows that

$$\begin{aligned}\alpha &= A, \\ \beta &= \frac{\alpha A - B}{2}, \\ \gamma &= \frac{\beta A - \alpha B + C}{3}, \\ \delta &= \frac{\gamma A - \beta B + \alpha C - D}{4}, \\ \epsilon &= \frac{\delta A - \gamma B + \beta C - \alpha D + E}{5} \\ &\text{etc.}\end{aligned}$$

We therefore obtain from these

$$\begin{aligned}\alpha &= \frac{n}{1-n}, \\ 2\beta &= \frac{\alpha n}{1-n} - \frac{nn}{1-nn}, \\ 3\gamma &= \frac{\beta n}{1-n} - \frac{\alpha n^2}{1-n^2} + \frac{n^3}{1-n^3}, \\ 4\delta &= \frac{\gamma n}{1-n} - \frac{\beta n^2}{1-n^2} + \frac{\alpha n^3}{1-n^3} - \frac{n^4}{1-n^4} \\ &\text{etc.}\end{aligned}$$

Now if however in place of α, β, γ etc. the previous values are successively

substituted, it follows

$$\begin{aligned}
\alpha &= \frac{n}{1-n}, \\
\beta &= \frac{n^3}{(1-n)(1-nn)}, \\
\gamma &= \frac{n^6}{(1-n)(1-nn)(1-n^3)}, \\
\delta &= \frac{n^{10}}{(1-n)(1-n^2)(1-n^3)(1-n^4)}, \\
\epsilon &= \frac{n^{15}}{(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5)} \\
&\text{etc.}
\end{aligned}$$

Thus we can understand that, in this case,

$$\begin{aligned}
\alpha &= A, \\
\beta &= AB, \\
\gamma &= ABC, \\
\delta &= ABCD, \\
\epsilon &= ABCDE \\
&\text{etc.}
\end{aligned}$$

21. The law according to which the values of the letters $\alpha, \beta, \gamma, \delta$ etc. have been found, whose truth is observed by expanding several formulae, is so far not apparent except by induction. So its truth may thus be securely confirmed, it will be convenient to elicit the same law of the progression in a totally different way, in which induction plays no part. Now, let it be proposed to investigate the values of the letters $\alpha, \beta, \gamma, \delta$ etc. which occur in the series

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5 + \text{etc.}$$

If as we had assumed initially

$$R = (1 + nz)(1 + n^2z)(1 + n^3z)(1 + n^4z) \cdots,$$

it should be noted that if nz is written in place of z , the expression which just now R was equal to is changed into the form

$$(1 + n^2z)(1 + n^3z)(1 + n^4z)(1 + n^5z) \cdots,$$

which multiplied by $1 + nz$ produces the prior expression. Therefore we can rightly conclude that if in the series

$$1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5 + \text{etc.}$$

we write nz in place of z , so that we have

$$1 + \alpha nz + \beta n^2 z^2 + \gamma n^3 z^3 + \delta n^4 z^4 + \epsilon n^5 z^5 + \text{etc.},$$

and we then multiply this expression by $1 + nz$, then the product, which will be

$$\begin{array}{cccccc} 1 & +\alpha nz & +\beta n^2 z^2 & +\gamma n^3 z^3 & +\delta n^4 z^4 & +\epsilon n^5 z^5 & +\text{etc.} \\ & +nz & +\alpha n^2 z^2 & +\beta n^3 z^3 & +\gamma n^4 z^4 & +\delta n^5 z^5 & +\text{etc.}, \end{array}$$

should be equal to that of the prior series

$$1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5 + \text{etc.}$$

But if we equate the coefficients of the corresponding terms, we will obtain the following determinations for α, β, γ etc.

$$\begin{aligned} \alpha &= \frac{n}{1-n} = \frac{n}{1-n}, \\ \beta &= \frac{\alpha n^2}{1-n^2} = \frac{n^3}{(1-n)(1-n^2)}, \\ \gamma &= \frac{\beta n^3}{1-n^3} = \frac{n^6}{(1-n)(1-n^2)(1-n^3)}, \\ \delta &= \frac{\gamma n^4}{1-n^4} = \frac{n^{10}}{(1-n)(1-n^2)(1-n^3)(1-n^4)} \\ &\text{etc.} \end{aligned}$$

22. In this way we have thus found convenient enough expressions for the sums of these series $\alpha, \beta, \gamma, \delta$ etc., from which in turn these series can be formed. Namely since this series proceed according to powers of n , this will arise if the expressions of these sums are expanded by division in the usual way into infinite series proceeding according to powers of n . Let this division be done, and it is clear that all the series $\alpha, \beta, \gamma, \delta$ etc. belong to a type which is commonly referred to by the name of recurrent series; they are such that any term can be determined from several of the preceding. So it will be clear how in each of these series any term is formed from the preceding, let us expand the denominators of these expressions found for the letters $\alpha, \beta, \gamma, \delta$ etc. by actual multiplication,

and having done this we will have

$$\begin{aligned}
\alpha &= \frac{n}{1-n}, \\
\beta &= \frac{n^3}{1-n-n^2+n^3}, \\
\gamma &= \frac{n^6}{1-n-n^2+n^4+n^5-n^6}, \\
\delta &= \frac{n^{10}}{1-n-n^2+2n^5-n^8-n^9+n^{10}}, \\
\epsilon &= \frac{n^{15}}{1-n-n^2+n^5+n^6+n^7-n^8-n^9-n^{10}+n^{13}+n^{14}-n^{15}}, \\
\zeta &= \frac{n^{21}}{1-n-n^2+n^5+2n^7-n^9-n^{10}-n^{11}-n^{12}+2n^{14}+n^{16}-n^{19}-n^{20}+n^{21}} \\
&\text{etc.}
\end{aligned}$$

And from these denominators we understand how in each series any term is composed from the preceding, if the precept by which these recurrent series are formed is called upon.

23. And from the form of the expressions found for the letters $\alpha, \beta, \gamma, \delta$ etc., by which each is the product of the preceding and some new factor, another method is deduced which is suitable enough for finding the next series from any series which has already been found. Thus, since the series $\alpha = \frac{n}{1-n}$ is a geometric progression

$$\alpha = n + n^2 + n^3 + n^4 + n^5 + n^6 + n^7 + \text{etc.},$$

from which comes the series β , if it is multiplied by $\frac{n^2}{1-n^2}$ or if it is multiplied by this geometric progression

$$n^2 + n^4 + n^6 + n^8 + n^{10} + n^{12} + n^{14} + \text{etc.}$$

Next, with the series β having been solidly found, if it is multiplied by

$$\frac{n^3}{1-n^3} = n^3 + n^6 + n^9 + n^{12} + n^{15} + n^{18} + \text{etc.},$$

series γ is produced. And this multiplied by

$$\frac{n^4}{1-n^4} = n^4 + n^8 + n^{12} + n^{16} + n^{20} + n^{24} + \text{etc.}$$

yields the series δ . And by so on multiplying each series of this order by a certain geometric progression the following series comes out. In this manner, these series can be continued as far as we want; and thus the above problem proposed by the Insightful Naudé is resolved.

24. And each of the series will be able to be easily continued by means of the preceding, if we consider the way in which the value of each of the letters $\alpha, \beta, \gamma, \delta$ etc. is determined from the preceding. Thus, since $\beta = \frac{\alpha n^2}{1-n^2}$, it will be $\beta = \beta nn + \alpha nn$; therefore if to the series β multiplied by nn is added the series α multiplied by nn , the series β will be created. Then, since it is clear that the first term of the series β is n^3 , let us put

$$\beta = \mathfrak{a}n^3 + \mathfrak{b}n^4 + \mathfrak{c}n^5 + \mathfrak{d}n^6 + \mathfrak{e}n^7 + \mathfrak{f}n^8 + \mathfrak{g}n^9 + \text{etc.}$$

and it will be

$$\begin{array}{cccccccc} \beta n^2 & = & & + \mathfrak{a}n^5 & + \mathfrak{b}n^6 & + \mathfrak{c}n^7 & + \mathfrak{d}n^8 & + \mathfrak{e}n^9 & + \text{etc.}, \\ \alpha n^2 & = & n^3 & + n^4 & + n^5 & + n^6 & + n^7 & + n^8 & + n^9 & + \text{etc.} \end{array}$$

Now with the terms equated, because $\beta = \beta nn + \alpha nn$ we will have

$$\begin{array}{ll} \mathfrak{a} = 1, & \mathfrak{e} = \mathfrak{c} + 1 = 3, \\ \mathfrak{b} = 1, & \mathfrak{f} = \mathfrak{d} + 1 = 3, \\ \mathfrak{c} = \mathfrak{a} + 1 = 2, & \mathfrak{g} = \mathfrak{e} + 1 = 4, \\ \mathfrak{d} = \mathfrak{b} + 1 = 2, & \mathfrak{h} = \mathfrak{f} + 1 = 4 \end{array}$$

etc.

In a similar way, since $\gamma = \frac{\beta n^3}{1-n^3}$ or $\gamma = \gamma n^3 + \beta n^3$, the series γ will be formed from the series β , and in turn the series δ is produced from the series γ , by means of the equation $\delta = \delta n^4 + \gamma n^4$; and all the following will be dispatched likewise.

25. Because in the expression

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

we have found the values of the letters $\alpha, \beta, \gamma, \delta$ etc. and it is

$$R = (1 + nz)(1 + n^2z)(1 + n^3z)(1 + n^4z) \dots,$$

this product, apparently

$$(1 + nz)(1 + n^2z)(1 + n^3z)(1 + n^4z) \dots,$$

will be converted from the infinite factors into this series proceeding according to powers of z

$$1 + \frac{nz}{1-n} + \frac{n^3z^3}{(1-n)(1-n^2)} + \frac{n^6z^3}{(1-n)(1-n^2)(1-n^3)} + \frac{n^{10}z^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)} + \text{etc.}$$

And from §10 the hyperbolic logarithm of the sum of this series will be

$$= \frac{nz}{1-n} - \frac{nnz^2}{2(1-n^2)} + \frac{n^3z^3}{3(1-n^3)} - \frac{n^4z^4}{4(1-n^4)} + \text{etc.}$$

Or if k is written for the number whose logarithm = 1, it will be

$$k^{\frac{nz}{1-n} - \frac{n^2z^2}{2(1-n^2)} + \frac{n^3z^3}{3(1-n^3)} - \frac{n^4z^4}{4(1-n^4)} + \text{etc.}} = R$$

or this exponential expression is equal to the sum of series which we made changed into the value of R .

26. Truly to turn the proposed problem, which is to define how many different ways a given number m can be partitioned into μ integral parts that are unequal to each other, let us indicate this number of ways which we are seeking by the notation

$$m^{(\mu)i},$$

which from now on will indicate to us the number of ways in which the number m can be produced from the addition of μ mutually unequal integral numbers; and for denoting the inequality of these parts we have adjoined the letter i above, which will be omitted if the question takes the form of finding altogether the number of ways in which the given number m can be distributed into μ parts, either equal or unequal. After this, a solution to the problem can easily be shown

27. Thus this number of ways $m^{(\mu)i}$ will be the coefficient of the power n^m in the series $\alpha, \beta, \gamma, \delta, \epsilon$ etc., which first at α are numbered as far as μ contains unities. The sum of this series is

$$= \frac{n^{\frac{\mu(\mu+1)}{1 \cdot 2}}}{(1-n)(1-n^2)(1-n^3)(1-n^4) \cdots (1-n^\mu)}$$

and hence the general term of the series which arises from this form is $= m^{(\mu)i} n^m$. Moreover, the general term of the series which arises from the form

$$\frac{n^{\frac{\mu(\mu-1)}{1 \cdot 2}}}{(1-n)(1-n^2)(1-n^3)(1-n^4) \cdots (1-n^\mu)}$$

will be $= m^{(\mu)i} n^{m-\mu}$ or for the same power of n the general term will be $= (m+\mu)^{(\mu)i} n^m$. The prior expression is subtracted from the latter, and the general term of the remaining expression

$$\frac{n^{\frac{\mu(\mu-1)}{1 \cdot 2}}}{(1-n)(1-n^2)(1-n^3)(1-n^4) \cdots (1-n^{\mu-1})}$$

will be $= n^m((m+\mu)^{(\mu)i} - m^{(\mu)i})$; moreover the general term of this series is $m^{(\mu-1)i} n^m$, whence we will have

$$m^{(\mu-1)i} = (m+\mu)^{(\mu)i} - m^{(\mu)i},$$

from which we arrive at the rule that

$$(m+\mu)^{(\mu)i} = m^{(\mu)i} + m^{(\mu-1)i},$$

by means of which, if the number of different ways the number m can be distributed into μ and $\mu - 1$ unequal parts were known, by adding these two numbers would follow the number of ways in which the larger number $m + \mu$ can be distributed into μ unequal parts. And thus the resolution of more difficult cases is reduced to simpler ones, and with these known finally to the simpler ones; it is of course clear that if $m < \frac{\mu\mu+\mu}{2}$ then $m^{(\mu)i} = 0$, and if $m = \frac{\mu\mu+\mu}{2}$ then it will be $m^{(\mu)i} = 1$.

28. Since the formula $m^{(\mu)i}n^m$ is the general term of the expression

$$\frac{n^{\frac{\mu(\mu+1)}{2}}}{(1-n)(1-n^2)(1-n^3)\cdots(1-n^\mu)},$$

let us see what kind of series this expression

$$\frac{1}{(1-n)(1-n^2)(1-n^3)\cdots(1-n^\mu)},$$

produces if expanded and arranged according to the dimensions of n . Let us put it to produce this series

$$1 + pn + qn^2 + rn^3 + sn^4 + tn^5 + \text{etc.},$$

from whose generation it is clear that the coefficient of any power of n shows how many different ways the exponent of n can be produced by addition from the given numbers

$$1, 2, 3, 4, 5, 6, \dots, \mu;$$

and here neither is a certain number of parts prescribed from which it is formed, nor is the condition put that the must be unequal to each other. Therefore the expression $m^{(\mu)i}$ will indicate altogether how many ways the number $m - \frac{\mu(\mu+1)}{2}$ can be produced by addition from the numbers $1, 2, 3, 4, 5, \dots, \mu$. Thus if one searches for how many different ways the number 50 can be distributed into 7 unequal parts, because $m = 50$ and $\mu = 7$ the question is thus reduced to investigating how many different ways the number $50 - 28$ or 22 can arise by addition from the seven numbers $1, 2, 3, 4, 5, 6, 7$. With this understood, both varieties of this question can be resolved in a single effort.

29. With the letters $\alpha, \beta, \gamma, \delta$ etc. defined for the case where we have assumed the geometric progression n, n^2, n^3, n^4, n^5 etc. in place of the letters a, b, c, d etc., order requires that we also inquire into the values of the third order $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$, etc. But we have employed the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., with equal values, in the series $\frac{1}{R}$ and $\frac{1}{S}$; for we assumed above (§11) that

$$\frac{1}{S} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \mathfrak{E}z^5 + \text{etc.}$$

and

$$\frac{1}{R} = 1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \mathfrak{E}z^5 + \text{etc.}$$

where the original values taken for R and S were

$$R = (1 + nz)(1 + n^2z)(1 + n^3z)(1 + n^4z) \text{ etc.},$$

$$S = (1 - nz)(1 - n^2z)(1 - n^3z)(1 - n^4z) \text{ etc.}$$

It is apparent hence that the series $\frac{1}{S} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}$ will arise if these innumerable geometric progressions are multiplied by each other

$$\begin{aligned} \frac{1}{1 - nz} &= 1 + nz + n^2z^2 + n^3z^3 + n^4z^4 + \text{etc.}, \\ \frac{1}{1 - n^2z} &= 1 + n^2z + n^4z^2 + n^6z^3 + n^8z^4 + \text{etc.}, \\ \frac{1}{1 - n^3z} &= 1 + n^3z + n^6z^2 + n^9z^3 + n^{12}z^4 + \text{etc.}, \\ \frac{1}{1 - n^4z} &= 1 + n^4z + n^8z^2 + n^{12}z^3 + n^{16}z^4 + \text{etc.} \\ &\text{etc.} \end{aligned}$$

On the other hand, by putting $-z$ in place of z the series $\frac{1}{R}$ follows in a similar way.

30. From the generation of these series it is clear that:

$$\text{I. } \mathfrak{A} = n + n^2 + n^3 + n^4 + n^5 + \text{etc.},$$

which is a geometric progression where all powers of n are multiplied by the coefficient $+1$.

$$\text{II. } \mathfrak{B} = n^2 + n^3 + 2n^4 + 2n^5 + 3n^6 + 3n^7 + 4n^8 + 4n^9 + \text{etc.},$$

in which the coefficient of each power of n contains as much unities as there are different ways in which the exponent of n can be partitioned into two parts, either equal or unequal. Thus the coefficient of the power n^8 is 4, because 8 can be partitioned into 2 parts in four ways

$$8 = 1 + 7, \quad 8 = 2 + 6, \quad 8 = 3 + 5, \quad 8 = 4 + 4.$$

$$\text{III. } \mathfrak{C} = n^3 + n^4 + 2n^5 + 3n^6 + 4n^7 + 5n^8 + 7n^9 + \text{etc.},$$

in which the coefficient of each power of n contains as many unities as there are different ways in which the exponent of n can be distributed into three parts, either equal or unequal. Thus n^9 has the coefficient 7, because 7 permits its separation into three parts in 9 ways:

$$\begin{aligned} 9 &= 1 + 1 + 7, & 9 &= 1 + 4 + 4, \\ 9 &= 1 + 2 + 6, & 9 &= 2 + 2 + 5, \\ 9 &= 1 + 3 + 5, & 9 &= 2 + 3 + 4, \end{aligned}$$

$$9 = 3 + 3 + 3.$$

$$\text{IV. } \mathfrak{D} = n^4 + n^5 + 2n^6 + 3n^7 + 5n^8 + 6n^9 + 9n^{10} + \text{etc.},$$

where the coefficient of any power of n contains as many unities as there are different ways in which the exponent of n can be resolved into four parts, either equal or unequal. And there is a similar rule for the following series which are found for the letters $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}$ etc.

31. The other problem which the Insightful Naudé proposed to me along with the preceding can thus also be resolved by means of these series. It can be expressed thus:

To find how many different way a given number m can be partitioned into μ parts, either equal or unequal, or to find how many different ways a given number m can be produced by addition from μ integral numbers, either equal or unequal.

The difference between this problem and the preceding is that in the preceding, the partition was restricted just to parts that were mutually unequal, while here equal parts are also allowed. For expressing the number of all these ways in this problem, the search will use this form

$$m^{(\mu)},$$

which declares namely how many different ways the number m can be partitioned into μ integral parts, with equality of some not excluded; for the letter i which was previously affixed to the above sign (μ) , which indicated unequal parts, is omitted here.

32. The solution of this problem can be reduced thus to the formation of the series $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc.; we have already shown above (§5) the values of these letters can be defined from the already known values of the letters $\alpha, \beta, \gamma, \delta$ etc. Though this method is general and attacks the very nature of the problem, it could however seem that the law by which these values proceed is not clear enough to the eyes. Therefore I shall investigate the values of the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc. in this case by a means similar to what I used above (§21).

Since it is

$$\frac{1}{S} = \frac{1}{(1 - nz)(1 - n^2z)(1 - n^3z)(1 - n^4z) \text{ etc.}},$$

it is clear that writing nz in place of z in this form will produce this form

$$\frac{1}{(1 - n^2z)(1 - n^3z)(1 - n^4z)(1 - n^5z) \text{ etc.}}.$$

But the prior form $\frac{1}{S}$ is turned into this if it is multiplied by $1 - nz$. Whence, since we have assumed that

$$\frac{1}{S} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \mathfrak{E}z^5 + \text{etc.},$$

we put here nz in place of z and we will have

$$1 + \mathfrak{A}nz + \mathfrak{B}n^2z^2 + \mathfrak{C}n^3z^3 + \mathfrak{D}n^4z^4 + \text{etc.}$$

Now let us multiply the first series $\frac{1}{S}$ by $1 - nz$

$$\begin{array}{cccccc} 1 & +\mathfrak{A}z & +\mathfrak{B}z^2 & +\mathfrak{C}z^3 & +\mathfrak{D}z^4 & +\text{etc.} \\ -nz & -\mathfrak{A}nz^2 & -\mathfrak{B}nz^3 & -\mathfrak{C}nz^4 & -\text{etc.} & \end{array}$$

Since this form should be equal to the previous, it will be

$$\begin{aligned} \mathfrak{A} &= \frac{n}{1-n} = \frac{n}{1-n}, \\ \mathfrak{B} &= \frac{\mathfrak{A}n}{1-n^2} = \frac{n^2}{(1-n)(1-n^2)}, \\ \mathfrak{C} &= \frac{\mathfrak{B}n}{1-n^3} = \frac{n^3}{(1-n)(1-n^2)(1-n^3)}, \\ \mathfrak{D} &= \frac{\mathfrak{C}n}{1-n^4} = \frac{n^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)} \\ &\text{etc.} \end{aligned}$$

33. Hence a new relation is perceived here between the values of the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. and the values of the letters $\alpha, \beta, \gamma, \delta$ etc. It is noteworthy that these values do not disagree with each other. For collecting §21 it is understood to be

$$\begin{aligned} \alpha &= \mathfrak{A}, \\ \beta &= n\mathfrak{B}, \\ \gamma &= n^3\mathfrak{C}, \\ \delta &= n^6\mathfrak{D}, \\ \epsilon &= n^{10}\mathfrak{E} \\ &\text{etc.} \end{aligned}$$

Thus it is clear from the rule of the coefficients that the series $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. agree completely with the series $\alpha, \beta, \gamma, \delta$ etc. except for the sole difference being in the exponents of n . Indeed in the series \mathfrak{A} the exponents are also equal to the exponents in the series α , but in the series \mathfrak{B} the exponents are one less than the exponents in the series β , in the series \mathfrak{C} the exponents are three less than the exponents in the series γ , and the differences proceed according to the trigonal numbers and so on.

34. Therefore by the series $\alpha, \beta, \gamma, \delta$ etc. which we showed above how to form and by which the first problem of Naudé was resolved, the latter problem proposed by Naudé can simultaneously be resolved, so that its solution is reduced to the solution of the first. Namely it will be

$$\begin{aligned} m^{(1)} &= m^{(1)i}, \\ m^{(2)} &= (m+1)^{(2)i}, \\ m^{(3)} &= (m+3)^{(3)i}, \\ m^{(4)} &= (m+6)^{(4)i} \end{aligned}$$

and generally

$$m^{(\mu)} = \left(m + \frac{\mu(\mu-1)}{2} \right)^{(\mu)i}$$

and in turn

$$m^{(\mu)i} = \left(m - \frac{\mu(\mu-1)}{2} \right)^{(\mu)}.$$

Furthermore, because we have also found that

$$(m+\mu)^{(\mu)i} = m^{(\mu)i} + m^{(\mu-1)i},$$

and reducing this to the present case it will be

$$\left(m - \frac{\mu(\mu-3)}{2} \right)^{(\mu)} = \left(m - \frac{\mu(\mu-1)}{2} \right)^{(\mu)} + \left(m - \frac{(\mu-1)(\mu-2)}{2} \right)^{(\mu-1)}$$

or for convenience

$$m^{(\mu)} = (m-\mu)^{(\mu)} + (m-1)^{(\mu-1)}.$$

The series for the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. are easily formed from this property, and so the latter problem is resolved.

35. For an example of this problem that Insightful Man presented the question of determining in how many different ways the number 50 can be separated into exactly seven parts, either equal or unequal. This question can hence be reduced to the first problem, with $m = 50$ and $\mu = 7$, if we search for how many different ways the number $50 + 21$ or 71 can be separated into seven unequal parts. This in fact can be done in 8946 different ways. Indeed besides this, the same number 8946 indicates (§28) how many different ways $71 - 28 = 43$ can be produced by addition from the numbers 1, 2, 3, 4, 5, 6, 7. And generally the number of ways $m^{(\mu)}$ in which the number m is resolved into μ parts, either equal or unequal, also shows how many different ways the number $m - \mu$ can be produced by addition from the particular numbers

$$1, 2, 3, 4, 5, \dots, \mu.$$

36. At the end of this paper there is a noteworthy observation to make, which however I have not yet been able to demonstrate with geometric rigor. Namely I have observed that if the infinitely many factors of the product

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.},$$

are expanded by actual multiplication, they produce this series

$$1 - n - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - n^{40} + n^{51} + \text{etc.},$$

where only those those powers of n occur whose exponents are contained in the form $\frac{3xx \pm x}{2}$. And if x is an odd number, the powers of n , which are $n^{\frac{3xx \pm x}{2}}$, will have the coefficient -1 , while if x is an even number then the powers $n^{\frac{3xx \pm x}{2}}$ will have the coefficient $+1$.

37. It is also worth noting that the reciprocal series of this, which arises from the expansion of this fraction

$$\frac{1}{(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.}},$$

yields namely the recurrent series

$$1 + 1n + 2n^2 + 3n^3 + 5n^4 + 7n^5 + 11n^6 + 15n^7 + 22n^8 + \text{etc.}$$

Of course this series multiplied by the above series

$$1 - n - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - \text{etc.}$$

produces unity. And in the first series the coefficient of any power of n contains as many unities as there are different ways in which the exponent of n can be distributed into parts; thus 5 can be resolved in seven ways into parts, as

$$\begin{array}{lll} 5 = 5, & 5 = 3 + 2, & 5 = 2 + 2 + 1, \\ 5 = 4 + 1, & 5 = 3 + 1 + 1, & 5 = 2 + 1 + 1 + 1, \\ & 5 = 1 + 1 + 1 + 1 + 1; \end{array}$$

where namely neither the number of parts or inequalities are prescribed.